

Different Volume Computation Methods of Graph Polytopes

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Abstract. The aim of this work is to introduce several different volume computation methods of the graph polytope associated with various type of finite simple graphs. Among them, we obtained the recursive volume formula (RVF) that is fundamental and most useful to compute the volume of the graph polytope for an arbitrary finite simple graph.

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1. Introduction

Bóna, Ju and Yoshida [2] enumerated certain weighted graphs with the following conditions : For a given positive integer k , a nonnegative integer n and a simple graph $G = (VG, EG)$ with $VG = [k]$, where $[k] := \{1, 2, \dots, k\}$ and $[k]_* := [k] \cup \{0\}$, we consider the set

$$W(n; G) := \{\alpha = (n_1, \dots, n_k) \in ([n]_*)^k \mid ij \in EG \Rightarrow n_i + n_j \leq n\}.$$

We call α satisfying the conditions in the set given above (*vertex- weighted graph*). In fact, the number of weighted graphs is given by an Ehrhart polynomial of some convex polytope in a unit k -hypercube. Such a convex polytope is determined uniquely for a given finite simple graph as follows: Let $G = (VG, EG)$ be a simple graph with $VG = [k]$. Then the **graph polytope** $P(G)$ associated with the graph G is defined as

$$P(G) := \{(x_1, x_2, \dots, x_k) \in [0, 1]^k \mid ij \in EG \Rightarrow x_i + x_j \leq 1\}$$

Our main concerns in this article are the volume computation results of graph polytope associated with many types of graphs using several different methods. In order to obtain the volume of the graph polytope we need a certain

kernel function $K : [0, 1]^2 \rightarrow \mathbb{R}$ defined by the following:

$$K(s, t) := \begin{cases} 1, & s + t \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Then the volume $\text{vol}(G)$ of the polytope $P(G)$ is

$$\text{vol}(G) = \int_{Q_n} H(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n,$$

where $Q_n = [0, 1]^n$ is the n -dimensional unit hypercube,

$$H(x_1, x_2, \dots, x_n) = \prod_{ij \in EG} K(x_i, x_j).$$

From now on, all graphs we mentioned will be finite simple graphs and all polytopes are convex. The volume of graph polytope $P(G)$ will be denoted by $\text{vol}(G)$ rather than $\text{vol}(P(G))$.

Chapter 2 introduces a recursive volume formula for the volume of the graph polytope and volume formulae for the graph polytope associated with various types of graphs. Chapter 3 describes the graph joins and the corresponding volume formula. Volume of the graph polytope associated with the bipartite graph with certain symmetry is dealt in chapter 4. In chapter 5 we use the operator theory to find values for interesting series. In the last chapter we mentioned another way to get the volume of graph polytopes, which uses Ehrhart polynomial and series.

2. Recursive Volume Formula

One of the key and most fundamental techniques is the *recursive volume formula*, or RVF. It is also useful. Next two lemmas will be used to prove the RVF, and can be shown easily. Applications of RVF are discussed.

Lemma 2.1. (polytope partitioning) *Let P be a polytope containing a point x and*

$d(x, F) := \inf\{d(x, y) | y \in F\}$. Then

$$\text{vol}(G) = \sum \frac{d(x, F) \text{vol}(F)}{n}, \quad (2.1)$$

where the sum runs over all facets of $P(G)$.

Lemma 2.2. *If a graph G has no isolated vertex, then the graph polytope $P(G)$ has no facet of the form $x_i = 1$. In other words, $P(G)$ is only composed of facets of form $x_k = 0$ or $x_i + x_j = 1$ for $ij \in E$.*

Theorem 2.3. (RVF) *Let $G = (VG, EG)$ be a graph with the vertex set $VG = [n]$ and having no isolated vertex. Then*

$$\text{vol}(G) = \sum_{i=1}^n \frac{\text{vol}(G - i)}{2n}, \quad (2.2)$$

where $G - i$ is the graph with the vertex set $[n] \setminus i$ and, accordingly, with the inherited edge set in the original edge set EG .

Proof: Let $x = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in P(G)$. Then $d(x, F) = \frac{1}{2}$ for the facet F in the hyperplane $x_i = 0$ (hence $F = P(G - i)$ in the hyperplane.) and $d(x, F) = 0$ for all other facets F since $x \in F$. Note that $\text{vol}(G - i)$ is the $(n - 1)$ -dimensional volume of the graph polytope $P(G - i)$. By the Lemmas 2.1 and 2.2 we have the desired formula (2.2). \square

Corollary 2.4 (path). *Let $L_n = ([n], EL_n)$ be the path with $EL_n := \{i(i + 1)|i \in [n - 1]\}$. Then*

$$\text{vol}(L_n) = \frac{E_n}{n!},$$

where E_n is the n -th Euler number. Hence its generating function is:

$$\sum_{n \geq 0} \text{vol}(L_n) x^n = \sec x + \tan x.$$

Proof: Let $\alpha_n = n! \text{vol}(L_n)$. For convenience, we define $\alpha_0 := 0! \text{vol}(L_0) = 1$. It is obvious that $\alpha_1 = 1 = E_1$.

$$\begin{aligned} \text{vol}(L_{n+1}) &= \frac{\sum_{i=0}^n \text{vol}(L_i \cup L_{n-i})}{2(n+1)} \\ &= \frac{\sum_{i=0}^n \text{vol}(L_i) \text{vol}(L_{n-i})}{2(n+1)} \end{aligned} \tag{2.3}$$

by RVF.

$$\begin{aligned} 2\alpha_{n+1} &= 2((n+1)!) \text{vol}(L_{n+1}) \\ &= n! \sum_{i=0}^n \text{vol}(L_i) \text{vol}(L_{n-i}) \text{ (by the formula (2.3))} \\ &= \sum_{i=0}^n \binom{n}{i} i! \text{vol}(L_i) (n-i)! \text{vol}(L_{n-i}) \\ &= \sum_{i=0}^n \binom{n}{i} \alpha_i \alpha_{n-i} \text{ for } n \geq 1. \end{aligned}$$

This says that α_n satisfies the same recurrence relation as E_n 's together with $E_0 = 1$. Hence we get the first conclusion $\text{vol}(P(L_n)) = \frac{E_n}{n!}$. Since exponential generating function of the Euler number is $\sec x + \tan x$, the second conclusion follows. \square

Corollary 2.5 (cycle). *Let $C_n = ([n], E_n)$ be the path with $E_n := \{i(i + 1)|i \in [n](n + 1 := 1)\}$. Then*

$$\text{vol}(C_n) = \frac{E_{n-1}}{2((n-1)!)}$$

Hence its generating function is:

$$\sum_{n \geq 1} \text{vol}(C_n) x^n = \frac{x(\sec x + \tan x)}{2}.$$

Proof: Removing an edge in the cycle C_n results in a path L_{n-1} . Hence RVF and Corollary 2.4 imply both of conclusions. \square

Corollary 2.6 (complete graph). Let $K_n = ([n], E_n)$ be the path with $E_n := \{ij | i, j \in [n]\}$. Then

$$\text{vol}(K_n) = 2^{1-n}.$$

Hence its generating function is:

$$\sum_{n \geq 1} \text{vol}(K_n) x^n = \frac{2x}{2-x}.$$

Proof: $\text{vol}(K_n) = \frac{n \text{vol}(K_{n-1})}{2n} = \frac{\text{vol}(K_{n-1})}{2}$. Since $\text{vol}(K_1) = 1$, the conclusion follows easily. \square

Corollary 2.7 (complete bipartite graph). Let $K_{s,t} = ([s] \cup [t], E_{st})$ be the path with $E_{st} := \{ij | i \in [s], j \in [t]\}$. Then

$$\text{vol}(K_{s,t}) = \frac{1}{\binom{s+t}{s}}. \quad (2.4)$$

Proof: Since $\text{vol}(K_{s,0}) = 0 = \text{vol}(K_{0,t})$, and

$$\text{vol}(K_{s,t}) = \frac{s \text{vol}(K_{s-1,t}) + t \text{vol}(K_{s,t-1})}{2(s+t)},$$

by induction, we have the required formula. \square

3. Volume of the graph joins

Join of graphs H and K is the graph $G = (VG, EG)$ where $VG = VH \cup VK$ and $EG = EH \cup EK \cup \{xy | x \in VH, y \in VK\}$. We denote the join of graphs H and K simply by $H+K$. Obviously, the associative law holds. So we can define $nG := G+G+\dots+G$ (add n times). The question is that how we can calculate the volume of $G+H$. Note that $K_{m+n} = K_m + K_n$, $K_{m,n} = D_m + D_n$, where D_n is a null graph with n vertices.

Definition 3.1 (sliced volume). Let $G = ([n], EG)$ be a graph and $a \in [0, 1]$.

$$\text{vol}(G, a) := \int_{[0,a]^n} \prod_{ij \in EG} K(x_i, x_j) dx,$$

where $K(\cdot, \cdot)$ is defined as:

$$K(s, t) := \begin{cases} 1, & s + t \leq a \\ 0, & \text{elsewhere.} \end{cases}$$

It is obvious that $\text{vol}(G) = \text{vol}(G, 1)$. Next theorem gives us a volume formula for the graph polytope associated with the joined graph.

Theorem 3.2. *The volume of the graph polytope associated with the graph $G + H$ is:*

$$\text{vol}(G + H, c) = \int_0^c \int_0^c \left(\frac{d}{ds} \text{vol}(G, s) \right) \left(\frac{d}{dt} \text{vol}(H, t) \right) K(s, t) ds dt. \quad (3.1)$$

Proof:

$$\text{vol}(G + H, c)$$

$$\begin{aligned} &= \int_{[0, c]^{|V(G+H)|}} \prod_{ij \in E(G+H)} K(x_i, x_j) dx \quad (x = (x_1, x_2, \dots, x_{|V(G+H)|})) \\ &= \int_{[0, c]^{|V(G+H)|}} \left(\prod_{ij \in E(G)} K(x_i, x_j) \right) \left(\prod_{kl \in E(H)} K(x_k, x_l) \right) \\ &\quad K(\max\{x_1, x_2, \dots, x_{|V(G)|}\}, \max\{x_{|V(G)|+1}, x_{|V(G)|+2}, \dots, x_{|V(G+H)|}\}) dx \\ &\quad (\text{let } s := \max\{x_1, x_2, \dots, x_{|V(G)|}\}, t := \max\{x_{|V(G)|+1}, x_{|V(G)|+2}, \dots, x_{|V(G+H)|}\}) \\ &= \int_0^c \int_0^c \left(\frac{d}{ds} \int_{[0, s]^{|V(G)|}} \prod_{ij \in E(G)} K(x_i, x_j) dx_G \right) \\ &\quad \left(\frac{d}{dt} \int_{[0, t]^{|V(H)|}} \prod_{kl \in E(H)} K(x_k, x_l) dx_H \right) K(s, t) ds dt \\ &\quad (\text{where } dx_G = dx_1 dx_2 \dots dx_{|V(G)|}, \quad dx_H = dx_{|V(G)|+1} dx_{|V(G)|+2} \dots dx_{|V(G+H)|}) \\ &= \int_0^c \int_0^c \left(\frac{d}{ds} \text{vol}(G, s) \right) \left(\frac{d}{dt} \text{vol}(H, t) \right) K(s, t) ds dt. \end{aligned}$$

□

Theorem 3.3. *Let $\frac{1}{2} \leq c \leq 1$. Then the sliced volume of the graph polytope $P(K_{m,n})$ is given by the formula:*

$$\text{vol}(P(K_{m,n}), c) = c^n(1-c)^m + m \sum_{i=0}^n \binom{n}{i} (-1)^i \frac{c^{m+i} - (1-c)^{m+i}}{m+i}.$$

Proof:

$$\begin{aligned} \text{vol}(K_{m,n}, c) &= \text{vol}(D_m + D_n, c) \\ &= \int_0^c \int_0^c \left(\frac{d}{ds} \text{vol}(D_m, s) \right) \left(\frac{d}{dt} \text{vol}(D_n, t) \right) K(s, t) ds dt \\ &= \int_0^c \int_0^c \frac{ds^m}{ds} \frac{dt^n}{dt} K(s, t) ds dt \\ &= \int_0^c \left(\frac{ds^m}{ds} \int_0^c K(s, t) \frac{dt^n}{dt} dt \right) ds \\ &= \int_0^c \left(\frac{ds^m}{ds} \int_0^{\min(1-s, c)} \frac{dt^n}{dt} dt \right) ds \\ &= \int_0^c \left(\frac{ds^m}{ds} (\min(1-s, c))^n \right) ds \\ &= \int_0^{1-c} c^n \frac{ds^m}{ds} ds + \int_{1-c}^c (1-s)^n \frac{ds^m}{ds} ds \\ &= c^n(1-c)^m + m \int_{1-c}^c s^{m-1} (1-s)^n ds \\ &= c^n(1-c)^m + m \sum_{i=0}^n \binom{n}{i} (-1)^i \int_{1-c}^c s^{m+i-1} ds \\ &= c^n(1-c)^m + m \sum_{i=0}^n \binom{n}{i} (-1)^i \frac{c^{m+i} - (1-c)^{m+i}}{m+i}. \end{aligned}$$

□

Corollary 3.4. *The volume of the graph polytope associated with the complete bipartite graph $K_{m,n}$ is:*

$$\text{vol}(K_{m,n}) = \sum_{i=0}^n \binom{n}{i} (-1)^i \frac{m}{m+i}. \quad (3.2)$$

Proof: Substitute $c = 1$ in the Theorem (3.3).

□

The following result is immediate by the third term from the bottom in the formula (3.3).

Corollary 3.5.

$$\sum_{i=0}^n \binom{n}{i} (-1)^i \frac{m}{m+i} = \frac{1}{\binom{m+n}{n}} = \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{n}{n+i}.$$

According to Rudin([5]), the *beta function* $B(r, s)$ is defined as:

$$B(r, s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)},$$

where $\Gamma(x)$ is the gamma function. Note that

$$\text{vol}(K_{m,n}) = mB(m, n+1) = nB(m+1, n)$$

from the formula 2.4 and the definition of beta function involving gamma function. Note that we can give another proof of above equality about beta function from the last integration expression in the proof.

Theorem 3.6 (multiple join of a graph). *Let G be a graph, $\frac{1}{2} \leq r \leq 1$ and $k := |VG|$. Then, for any positive integer n ,*

$$\frac{d}{dr}\text{vol}(nG, r) = n(1-r)^{k(n-1)} \frac{d}{dr}\text{vol}(G, r).$$

Proof:

$$\begin{aligned} \text{vol}(nG, r) &= \int_0^r \int_0^r \left(\frac{d}{ds}\text{vol}(G, s) \right) \left(\frac{d}{dt}\text{vol}((n-1)G, t) \right) K(s, t) ds dt \\ &= \int_0^r \frac{d}{ds}(\text{vol}(G, s)) \left(\int_0^{\min(1-s, r)} \frac{d}{dt}\text{vol}((n-1)G, t) dt \right) ds \\ &= \int_0^r \frac{d}{ds}(\text{vol}(G, s) \text{vol}((n-1)G, \min(1-s, r))) ds \\ &= (i) + (ii) + (iii) \end{aligned}$$

where

$$\begin{aligned} (i) &= \int_0^{1-r} \text{vol}((n-1)G, r) \frac{d}{ds}\text{vol}(G, s) ds \\ &= \text{vol}((n-1)G, r)(1-r)^k \quad (0 \leq 1-r \leq \frac{1}{2}) \\ (ii) &= \int_{1-r}^{1/2} \text{vol}((n-1)G, 1-s) \frac{d}{ds}\text{vol}(G, s) ds \\ &= \int_{1-r}^{1/2} \text{vol}((n-1)G, 1-s) |VG| s^{|VG|-1} ds \\ (iii) &= \int_{1/2}^r \text{vol}((n-1)G, 1-s) \frac{d}{ds}\text{vol}(G, s) ds \\ &= \int_{1/2}^r (1-s)^{(n-1)|VG|} \frac{d}{ds}\text{vol}(G, s) ds. \end{aligned}$$

Now, we denote $\frac{d}{dr} \text{vol}(nG, r)$ simply by a_n . Then

$$\begin{aligned} a_n &= \frac{d}{dr}((i) + (ii) + (iii)) \\ &= (1-r)^k a_{n-1} - k(1-r)^{k-1} \text{vol}((n-1)G, r) \\ &\quad + k(1-r)^{k-1} \text{vol}((n-1)G, r) + a_1(1-r)^{(n-1)k} \\ &= (1-r)^k a_{n-1} + (1-r)^{(n-1)k} a_1. \end{aligned}$$

Let $F(x, r) = \sum_{n \geq 1} a_n x^n$. Then, from the previous recursion formula we get

$$F(x, r) = \frac{xa_1}{(1 - (1-r)^k x)^2} = \sum_{n \geq 1} na_1(1-r)^{(n-1)k} x^n.$$

Hence,

$$\frac{d}{dr} \text{vol}(nG, r) = n(1-r)^{(n-1)k} \frac{d}{dr} \text{vol}(G, r).$$

□

Corollary 3.7. *For the value $\frac{1}{2} \leq r \leq 1$, we have*

$$\text{vol}(K_n, r) = 2^{1-n} - (1-r)^n \text{ and } \text{vol}(K_n) = 2^{1-n}. \quad (3.3)$$

Proof: Since $K_n = nD_1$, we have the formula (3.3) from the Theorem 3.6. □

Corollary 3.8. *For the value $\frac{1}{2} \leq r \leq 1$, we have*

$$\text{vol}(nD_k) = 2^{-kn} + n2^{-kn} \frac{1}{\binom{kn}{k}} \sum_{i=0}^{k-1} \binom{kn}{i}.$$

Proof:

$$\frac{d}{dt} \text{vol}(nD_k, t) = n(1-t)^{k(n-1)} \frac{d}{dt} \text{vol}(D_k, t) = knt^{k-1}(1-t)^{k(n-1)}.$$

$$\begin{aligned}
 \text{vol}(nD_k) &= 2^{-kn} + kn \int_{1/2}^1 t^{k-1} (1-t)^{k(n-1)} dt \\
 &= 2^{-kn} + kn \int_1^0 \left(-\frac{1}{2}\right) \left(1-\frac{t}{2}\right)^{k-1} \left(\frac{t}{2}\right)^{k(n-1)} dt \\
 &= 2^{-kn} + kn 2^{-kn} \int_0^1 t^{k(n-1)} (2-t)^{k-1} dt \\
 &= 2^{-kn} + kn 2^{-kn} \sum_{i=0}^{k-1} \binom{k-1}{i} \int_0^1 t^{k(n-1)} (1-t)^i dt \\
 &= 2^{-kn} + kn 2^{-kn} \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(k(n-1))! i!}{(k(n-1) + i + 1)!} \\
 &= 2^{-kn} + kn 2^{-kn} \sum_{i=0}^{k-1} \frac{1}{k \binom{kn}{k}} \binom{kn}{k-1-i} \\
 &= 2^{-kn} + kn 2^{-kn} \frac{1}{\binom{kn}{k}} \sum_{i=0}^{k-1} \binom{kn}{i}
 \end{aligned}$$

□

4. Volume of bipartite graphs

Definition 4.1. Let S_n be a set of all permutations of $[n]$ and $\sigma \in S_n$.

$$[0, 1]_\sigma^n := \{x = (x_1, x_2, \dots, x_n) \in [0, 1]^n \mid x_{\sigma(1)} \leq x_{\sigma(2)} \leq \dots \leq x_{\sigma(n)}\}.$$

Note that

$$[0, 1]^n = \bigcup_{\sigma \in S_n} [0, 1]_\sigma^n$$

and each intersection of two different $[0, 1]_\sigma^n$ has measure 0 so that for any measurable function f ,

$$\int_{[0, 1]^n} f dx = \sum_{\sigma \in S_n} \int_{[0, 1]_\sigma^n} f dx$$

and

$$\int_{[0, 1]_\sigma^n} f dx = \int_0^1 \left(\int_0^{x_{\sigma(n)}} \left(\int_0^{x_{\sigma(n-1)}} \dots \left(\int_0^{x_{\sigma(2)}} f dx_{\sigma(1)} \right) \dots dx_{\sigma(n-2)} \right) dx_{\sigma(n-1)} \right) dx_{\sigma(n)}.$$

Let $B = (VB, EB)$ be a bipartite graph with $VB = V_1 \cup V_2$, $V_1 = \{1, 2, \dots, n\}$, $V_2 = \{v_1, v_2, \dots, v_m\}$, $N_i = \{j \in V_1 \mid jv_i \in EB\}$.

Theorem 4.2. *The volume of the graph polytope associated with the bipartite graph b mentioned above is as follows:*

$$\text{vol}(B) = \sum_{\sigma \in S_n} \prod_{i=1}^n \frac{1}{i + \sum_{j=1}^i \alpha_{j,\sigma}},$$

where

$$\alpha_{i,\sigma} = \text{number of } (\{v_k \in V_2 | \sigma(i) \in N_k\} \setminus \cup_{j=1}^{i-1} \{v_k \in V_2 | \sigma(j) \in N_k\}).$$

which means the number of vertices in V_2 which the smallest among σ values of its neighbors is i .

Proof:

$$\begin{aligned} \text{vol}(B) &= \int_{[0,1]^n} \prod_{j=1}^m (1 - \max\{x_i \mid i \in N_j\}) dx \\ &= \sum_{\sigma \in S_n} \int_{[0,1]_\sigma^n} \prod_{j=1}^m (1 - \max\{x_i \mid i \in N_j\}) dx \\ &= \sum_{\sigma \in S_n} \int_{[0,1]_\sigma^n} \prod_{j=1}^m (\min\{1 - x_i \mid i \in N_j\}) dx \\ &= \sum_{\sigma \in S_n} \int_{[0,1]_{\text{rev}(\sigma)}^n} \prod_{j=1}^m \min\{x_i \mid i \in N_j\} dx \\ &= \sum_{\sigma \in S_n} \int_{[0,1]_\sigma^n} \prod_{j=1}^m \min\{x_i \mid i \in N_j\} dx \\ &= \sum_{\sigma \in S_n} \int_{[0,1]_\sigma^n} \prod_{i=1}^n x_{\sigma(i)}^{\alpha_{i,\sigma}} dx \\ &= \sum_{\sigma \in S_n} \int_0^1 (x_{\sigma(n)}^{\alpha_{n,\sigma}} \int_0^{x_{\sigma(n)}} [x_{\sigma(n-1)}^{\alpha_{n-1,\sigma}} \cdots \\ &\quad \int_0^{x_{\sigma(3)}} (x_{\sigma(2)}^{\alpha_{2,\sigma}} \int_0^{x_{\sigma(2)}} (x_{\sigma(1)}^{\alpha_{1,\sigma}}) dx_{\sigma(1)}) dx_{\sigma(2)} \cdots] dx_{\sigma(n-1)}) dx_{\sigma(n)} \\ &= \sum_{\sigma \in S_n} \prod_{i=1}^n \frac{1}{i + \sum_{j=1}^i \alpha_{j,\sigma}}. \end{aligned}$$

where $\text{rev}(\sigma)$ represents the opposite order of σ in the fifth term. \square

An *automorphism* of a simple graph $G = (VG, EG)$ is a permutation π of VG which has the property that uv is an edge of G if and only if $\pi(u)\pi(v)$ is an edge of G .

Theorem 4.3. Assume that the bipartite graph B is symmetric, by the sense that for any permutation π on V_1 , there exists a permutation σ such that the combination of π and σ induces an automorphism on G . Then,

$$\text{vol}(P(B)) = n! \prod_{i=1}^n \frac{1}{i + \sum_{j=1}^i \alpha_j}$$

Proof: The symmetry of the graph B implies that all $\alpha_{i,\sigma}$ s are same for different $\sigma \in S_n$. The conclusion follows from the Theorem 4.2. \square

Corollary 4.4. *Let B_n be the graph that is obtained from the complete bipartite graph $K_{n,n}$ by deleting n disjoint edges. Then,*

$$\text{vol}(B_n) = \left(1 + \frac{1}{n}\right) \frac{1}{\binom{2n}{n}}.$$

In particular, $\text{vol}(B_3) = \frac{1}{15}$. Note that the bipartite graph B_3 is the graph obtained from the 1-skeleton of the cube.

5. An Application related to the Operator Theory

We introduce here another interesting fact that uses the linear operator theory to obtain the value of a series described in the theorem below. One of the results related with the operator theory is the computation of the $\text{vol}(C_n)$, which is referred from Elkies [3]. We will restate the lemma regarding this.

We define $K : [0, 1]^2 \rightarrow \mathbb{R}$ by the following:

$$K(s, t) := \begin{cases} 1, & s + t \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Then the volume $\text{vol}(G)$ of the polytope $P(G)$ is

$$\text{vol}(G) = \int_{Q_n} H(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n,$$

where $Q_n = [0, 1]^n$ is the n -dimensional unit hypercube,

$$H(x_1, x_2, \dots, x_n) = \prod_{ij \in E} K(x_i, x_j).$$

We are interested in the computation of the volume of the polytope $P(G)$ for a given simple graph $G = (V, E)$.

We define \mathcal{K}_n inductively as in the following:

$$\mathcal{K}_1(t, s) := K(t, s),$$

and

$$\mathcal{K}_n(t, s) := \int_0^1 \mathcal{K}_1(t, x_1) \mathcal{K}_{n-1}(x_1, s) dx_1 \quad (n \geq 2).$$

Let $T : L^2(0, 1) \rightarrow L^2(0, 1)$ be a linear operator with kernel $\mathcal{K}_1(\cdot, \cdot)$ on $L^2(0, 1)$ defined by:

$$(Tg)(t) = \int_0^1 \mathcal{K}_1(t, s) g(s) ds = \int_0^{1-t} g(s) ds. \quad (5.1)$$

From the definition of \mathcal{K}_n we see that $\mathcal{K}_n(\cdot, \cdot)$ is the kernel function of the linear operator T^n as follows:

$$(T^n g)(t) = \int_0^1 \mathcal{K}_n(t, s) g(s) ds. \quad (5.2)$$

The next lemma gives the spectral decomposition of the linear operator T , and also of T^n . Its proof is immediate from the standard linear operator theory. (See also Elkies [3] or Hutson. et. al. [4].)

Lemma 5.1. *The linear operator T is compact and self-adjoint on $L^2(0, 1)$. Its eigen values are $\frac{2}{\pi(4k+1)}$ ($k \in \mathbf{Z}$); the corresponding eigenfunctions are $\cos(\pi(4k+1)/2)$. Moreover, The linear operator T^n is compact and self-adjoint on $L^2(0, 1)$. Its eigen values are $(\frac{2}{\pi(4k+1)})^n$ with same corresponding eigenfunctions $\cos(\pi(4k+1)/2)$. Each of its eigenvalues for T and T^n is simple.*

Our main goal here is to find the value of certain formula using the operator theory. In fact, it is the $\text{vol}(P(C_n))$ which is obtained from the RVF. By the simple calculations we can get the following formula from the definition of \mathcal{K}_n (see [3]):

$$\text{vol}(C_n) = \int_0^1 \mathcal{K}_n(t, t) dt. \quad (5.3)$$

It turned out that the right hand side of the formula (5.3) is the trace of a trace-class operator T^n over the diagonal, and is equal to

$$\sum_{k=-\infty}^{\infty} \frac{2^n}{(\pi(4k+1))^n}.$$

Note that this series is absolutely convergent for $n \geq 2$. As a summary we have a theorem:

Theorem 5.2. *For any integer $n \geq 2$ the following holds:*

$$\sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n} = \frac{\pi^n \text{vol}(C^n)}{2^n}.$$

□

For the case $n = 3$,

$$\begin{aligned} 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots + \frac{(-1)^m}{(2m+1)^3} + \cdots &= \frac{\pi^3 \text{vol}(C^3)}{8} \\ &= \frac{\pi^3 \text{vol}(K_3)}{8} \\ &= \frac{\pi^3 2^{-2}}{8} \\ &= \frac{\pi^3}{32}, \end{aligned}$$

meanwhile, for the case $n = 4$,

$$\begin{aligned} 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots + \frac{1}{(2m+1)^4} + \cdots &= \frac{\pi^4 \text{vol}(C^4)}{16} \\ &= \frac{\pi^4 \text{vol}(K_{2,2})}{16} \\ &= \frac{\pi^4}{16 \binom{2+2}{2}} \\ &= \frac{\pi^4}{96}. \end{aligned}$$

6. Concluding Remarks and Further Problems

In fact, we have another volume computation method which comes from the Ehrhart polynomial of $P(G)$. Let P be an integral convex polytope in \mathbb{R}^d . Then we call $L_P(t) = |tP \cap \mathbb{Z}^d|$ the *Ehrhart polynomials* of P . It is known that, for a given 0/1-polytope P ,

$$\text{vol}(P) = \lim_{t \rightarrow \infty} \frac{L_P(t)}{t^d}, \text{ where } d = \dim(P),$$

or

$$\frac{f(1)}{d!}, \text{ where } \sum_{t=0}^{\infty} L_P(t)x^t = \frac{f(x)}{(1-x)^{d+1}}.$$

(Refer [1] or [6] about this.) If G is a bipartite graph with n vertices, then its graph polytope $P(G)$ is a 0/1-polytope of dimension n . Hence, we can get the volume $\text{vol}(G)$ from the Ehrhart polynomial $L_{P(G)}(t)$, which we can get by using divided difference technique. (See Bóna et. al. [2] for details.)

References

- [1] M. Beck and R. Sinai, *Computing the Continuous Discretely*, Springer, 2007.
- [2] M. Bóna, H.-K. Ju and R. Yoshida, *On the enumeration of a certain weighted graphs*, Discrete Applied Math., 155(2007), 1481-1496.
- [3] N. Elkies, *On the sums $\sum_{k=-\infty}^{\infty} (4k+1)^{-n}$* , Amer. Math. Monthly, 110, no. 7(Aug.-Sep. 2003), 671-573.
- [4] V. Hutson, J. Pym and M. Cloud, *Applications of Functional Analysis and Operator Theory* (2nd ed.), Elsevier Science, 2005.
- [5] W. Rudin, *Principles of Mathematical Analysis* (3rd ed.), McGraw-Hill, 1976.
- [6] R. Stanley, *Enumerative Combinatorics*, vol.1(2nd ed.), Cambridge Univ. Press, 2012.

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